JOURNAL OF APPROXIMATION THEORY 52, 141–148 (1988)

Applications of the Proximity Map to Fixed Point Theorems in Hilbert Space

TZU-CHU LIN

Department of Mathematics, University of Wisconsin, Milwaukee, Wisconsin 53201, U.S.A.

AND

CHI-LIN YEN

Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan, Republic of China

Communicated by Paul G. Nevai

Received May 3, 1984

We extend Lin's result (*Canad. Math. Bull.* **22**, No. 4 (1979), 513–515) and Singh and Watsons' result (*J. Approx. Theory* **39** (1983), 72–76) to more general 1-set-contractive maps. This class of 1-set-contractive maps includes condensing (or densifying) maps and nonexpansive maps; it also includes other important maps such as semicontractive maps and LANE maps. As applications of our theorems, fixed point theorems are proved under various conditions. The main idea we use, is one due to Cheney and Goldstein (*Proc. Amer. Math. Soc.* **10** (1959), 448–450), that a proximity map in Hilbert space is nonexpansive.

1. INTRODUCTION AND PRELIMINARY DEFINITIONS

In [6], Fan proved the following theorem:

Let K be a nonempty compact convex set in a normed linear space X. For any continuous map f of K into X, there exists a point $u \in K$ such that

$$\|u - f(u)\| = d(f(u), K).$$

In [11], the first author proved that the above theorem is true for a continuous densifying map defined on a closed ball in a Banach space. Lin [11] also proved that the above theorem is true for a continuous densifying map defined on a closed convex subset of a Hilbert space. Recently, Singh and Watson [19] proved that the above theorem is true for a nonexpansive map defined on a closed convex subset of a Hilbert space.

0021-9045/88 $3.00

In this paper, we extend above results to more general 1-set-contractive maps. The class of 1-set-contractive maps includes densifying maps and nonexpansive maps. Besides, it also includes other important maps such as semicontractive maps and LANE maps which were not considered by Lin [11] and Singh and Watson [19]. The major idea we used is that the proximity map is nonexpansive in Hilbert space (due to Cheney and Goldstein [5]). We remark that the same idea was used by the first author in [11]. As applications of our theorems, various fixed point theorems are proved under various well-known conditions. Hence we extend these fixed point theorems, at least, in Hilbert space.

Now, we introduce our notations and definitions:

Let B be a nonempty bounded subset of a metric space X. We shall denote (after Kuratowski [10]) by $a(B)$ the infimum of the numbers r such that B can be covered by a finite number of subsets of X of diameter less than or equal to r.

Let S be a nonempty subset of X and let f be a map from S into X. If for every nonempty bounded subset B of S with $a(B)>0$, we have $a(f(B))<a(B)$, then f will be called densifying [7]. If there exists k, $0\leqslant k\leqslant 1$, such that for each nonempty bounded subset B of S we have $a(f(B))\leqslant ka(B)$, then f is called k-set-contractive [10].

Let X, Y be two normed linear spaces, S a nonempty subset of X, f a map from S into Y, f is called nonexpansive if for each x, $y\in S$, we have $\|f(x)-f(y)\|\leqslant\|x-y\|$. We remark that a nonexpansive map is also a 1-set-contractive map.

We also remark that a densifying map is also called a condensing (or more precisely set-condensing) map. It is closely related to the condensing (or more precisely ball-condensing) map developed by Sadovski [17] (e.g., see [13, 14]).

Let S be a nonempty subset of a normed linear space X. Then, for each x in X, define

$$d(x, S)=\inf_{y\in S}\|x-y\|$$

and

$$p_S(x)=\{y\in S \mid \|x-y\|=d(x, S)\}.$$

The set-valued map $p_S(x)$ is called the metric projection on S. If $p_S(x)$ is a single valued map, it is called a proximity map. The closed convex hull of S will be denoted by cl co S. The closure of S will be denoted by cl(S) or $\bar{S}$.

Let X be a Banach space, S a nonempty subset of X, f a map of S into X. Then f is said to be semicontractive [1] if there exists a map V of $S\times S$ into X such that $f(x)=V(x, x)$ for x in S, while:

(a) For each fixed x in S, $V(\cdot, x)$ is nonexpansive from S to X.

(b) For each fixed x in S, $V(x, \cdot)$ is complete continuous from S to X, uniformly for u in bounded subsets of S (i.e., if v_j converges weakly to v in S and $\{u_j\}$ is a bounded sequence in S, then $V(u_j, v_j) - V(u_j, v) \to 0$, strongly in S).

Let S be a nonempty closed bounded convex subset of a Banach space X. We say (after Nussbaum [12]) that a continuous map f of S into X is LANE (locally almost nonexpansive) if given $x \in S$ and $\varepsilon > 0$, there exists a weak neighborhood N_x of x in S (depending also on ε) such that $u, v \in N_x$, $\|f(u) - f(v)\| \leqslant \|u - v\| + \varepsilon$. A map g of S into X is called complete continuous (it is also called strongly continuous by some authors) if it maps weakly convergent sequences into strongly convergent sequences. If a sequence $\{u_j\}$ converges weakly to u_0 in S (denote by $u_j \rightharpoonup u_0$) and $(I-g)\, u_j$ converges strongly to w, we must have $(I-g)(u_0) = w$, then $I-g$ is called demiclosed.

Most definitions that we state above can be found in [13, 14, or 2]. For completeness, we state above definitions in details. For an expert on fixed point theory, our preliminaries may seem a little wordy. But for the majority of the readers—approximation theorists—our treatment may seem more appropriate and self-contained.

2. Main Results

Lemma 1. *Let S be a nonempty closed convex subset of a Banach space X, f a continuous 1-set-contractive map of S into S. Suppose that $f(S)$ is bounded and $(I-f)(S)$ is closed in X. Then f has a fixed point in S.*

This lemma is a special case of [16, Corollary 1.3]. If S is bounded instead of $f(S)$ is bounded, then the above lemma is well known (e.g., see [2, p. 230]). We remark that $f(S)$ is bounded if S is bounded.

Theorem 1. *Let S be a nonempty closed convex subset of a Hilbert space X, f a continuous 1-set-contractive map of S into X. Suppose that either (I-pof)(S) is closed in X or (I-pof) (cl co pof(S)) is closed in X, where p is the proximity map of X into S. If $f(S)$ is bounded, then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. By [5], p is nonexpansive in Hilbert space. Then pof is a continuous 1-set-contractive map of S into S and also of cl co pof(S) into cl co pof(S). Since $f(S)$ is bounded, pof(S) is also bounded. By Lemma 1, there exists a point u in S such that pof$(u) = u$. Hence

$$\|u - f(u)\| = \|p(f(u)) - f(u)\| = d(f(u), S).$$

COROLLARY 1 (Lin [11, Theorem 2]). *Let S be a nonempty closed convex subset of a Hilbert space X, f a continuous densifying map of S into X. If $f(S)$ is bounded, then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. Let p be the proximity map of X into S. Since p is nonexpansive [5] and f is continuous densifying, then pof is also a continuous densifying map of cl co pof(S) into cl co pof(S). From [15, Lemma 1], (I-pof) (cl co pof(S)) is closed in X. Since f is also a 1-set-contractive map, from Theorem 1, there exists a point u in S such that

$$\|u - f(u)\| = d(f(u), S).$$

COROLLARY 2 (Singh and Watson [19, Theorem 5]). *Let S be a nonempty closed convex subset of a Hilbert space X, f a nonexpansive map from S into X. If $f(S)$ is bounded, then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. Following the same proof as that of Corollary 1, pof is a nonexpansive and continuous 1-set-contractive map of cl co pof(S) into cl co pof(S). From [1], (I-pof) (cl co pof(S)) is closed in X. By Theorem 1, there exists a point u in S such that

$$\|u - f(u)\| = d(f(u), S).$$

Now we consider some maps which are neither continuous densifying nor nonexpansive. We prove that the above result is still true for these maps.

THEOREM 2. *Let S be a nonempty closed convex subset of a Hilbert space X, f a continuous semicontractive map of S into X. If $f(S)$ is bounded, then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. From [14, Lemma 3.2 and p. 338], f is a 1-set-contractive map. Since f is semicontractive, there exists a continuous map $V: S \times S \to X$ such that $f(x) = V(x, x)$ for $x \in S$, $V(\cdot, x)$ is a nonexpansive map of S into X and $V(x, \cdot)$ is a complete continuous map of S into X, uniformly for x in S. Since the proximity map p is nonexpansive from X to S, it is easily to see that poV has all the properties which V has. Therefore pof is a continuous semicontractive map. From [1], (I-pof)(cl co pof(S)) is closed in X. By Theorem 1, there exists a point u in S such that

$$\|u - f(u)\| = d(f(u), S).$$

COROLLARY 3. *Let S be a nonempty closed convex subset of a Hilbert space X, g a nonexpansive map of S into X, and h a complete continuous map* *of S into X. If $f = g + h$ and $f(S)$ is bounded, then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. Since $f = g + h$ is a semi-contractive map under the representation $V(u, v) = g(u) + h(v)$, this theorem just follows from Theorem 2.

Remark. A semicontractive map and a complete continuous map are also called a map of semicontractive type [14] and a strongly continuous map [15], respectively. We also remark that Corollary 2 can also be viewed as a special case of Corollary 3, if we set $h \equiv 0$.

THEOREM 3. *Let S be a nonempty, closed bounded convex subset of a Hilbert space X, f a* LANE *map of S into X. Then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. From [12], f is a 1-set-contractive map. Let p be the proximity map of X into S. Since f is a LANE map and p is a nonexpansive map, then pof is also a LANE map of S into S. From [12], I-pof is demiclosed. Now, we claim that $(I\text{-pof})(S)$ is closed. Let $y \in \text{cl}((I\text{-pof})(S))$, there exists a sequence $\{x_n\}$ in S such that $x_n\text{-pof}(x_n) \to y$. Since S is weakly compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x \in S$. Without loss of generality we assume that $x_n \rightharpoonup x$. By the demiclosedness of I-pof, then $x\text{-pof}(x) = y$ and $y \in (I\text{-pof})(S)$. Therefore $(I\text{-pof})(S)$ is closed in X. By Theorem 1, there exists a point u in S such that

$$\|u - f(u)\| = d(f(u), S).$$

THEOREM 4. *Let S be a nonempty closed bounded convex subset of a Hilbert space X, g a* LANE *map of S into X, and h a complete continuous map of S into X. If $f = g + h$, then there exists a point u in S such that*

$$\|u - f(u)\| = d(f(u), S).$$

Proof. From [14, Remark 3.7], f is also a LANE map. Hence the conclusion follows from Theorem 3.

3. APPLICATIONS TO FIXED POINT THEOREMS

THEOREM 5. *Let S be a nonempty closed convex subset of a Hilbert space X, f a continuous 1-set-contractive map of S into X. Suppose that either $(I\text{-pof})(S)$ is closed in X or $(I\text{-pof})(\text{cl co pof}(S))$ is closed in X, where* *p is the proximity map of X into S. If f(S) is bounded and f satisfies any one of the following conditions:*

(1) *For each $x \in S$, there is a number λ (real or complex, depending on whether the vector space X is real or complex) such that $|\lambda| < 1$ and $\lambda x + (1-\lambda) f(x) \in S$.*

(2) *If for $x \in S$ with $x \neq f(x)$, there exists y in $I_S(x) = \{x + c(z-x) |$ for some $z \in S$, some $c > 0\}$ such that*

$$\|y - f(x)\| < \|x - f(x)\|.$$

(3) *f is weakly inward (i.e., $f(x) \in \text{cl } I_S(x)$, for each $x \in S$).*

(4) *For any u on the boundary of S with $u = pof(u)$. that u is a fixed point of f.*

(5) *For each x on the boundary of S, $\|f(x) - y\| \leqslant \|x - y\|$ for some y in S.*

Then f has a fixed point in S.

Proof. Assume that f satisfies condition (1). By Theorem 1, there exists a point u in S such that

$$\|u - f(u)\| = d(f(u), S).$$

Suppose f has no fixed point in S, then $0 < \|u - f(u)\|$. To this u, there is a number λ such that $|\lambda| < 1$ and $\lambda u + (1-\lambda) f(u) = x \in S$. Therefore

$$0 < \|u - f(u)\| = d(f(u), S) \leqslant \|x - f(u)\|$$
$$= |\lambda| \, \|u - f(u)\| < \|u - f(u)\|,$$

which is a contradiction. Hence f has a fixed point in S.

Assume that f satisfies condition (2). By Theorem 1, there exists a point u in S such that

$$\|u - f(u)\| = d(f(u), S).$$

If $u \neq f(u)$, there exists y in $I_S(u)$ such that $\|y - f(u)\| < \|u - f(u)\|$. If $y \in S$, this contradicts the choice of u. Therefore $y \notin S$, and there exists $z \in S$ such that $y = u + c(z - u)$ for some $c > 1$. That is $z = (1/c)\, y + (1 - (1/c))\, u = (1-\beta)\, y + \beta u$, where $\beta = 1 - (1/c)$, $0 < \beta < 1$. Hence

$$\|z - f(u)\| = \|(1-\beta)\, y + \beta u - f(u)\| \leqslant (1-\beta)\, \|y - f(u)\| + \beta\, \|u - f(u)\|$$
$$< (1-\beta)\, \|u - f(u)\| + \beta\, \|u - f(u)\| = \|u - f(u)\|,$$

which contradicts the choice of u. Therefore $u = f(u)$.

It is not hard to show that if f satisfies the condition (3), then f also satisfies the condition (2).

For the conditions (4) and (5), using Theorem 1 and following the same proof as that of [19, Theorems 6, 7], we can conclude that f has a fixed point in S.

Remark. The conditions (1) and (2) were first considered by Fan [6] and Browder [3] in an attempt to extend fixed point theorems to inward and weakly inward maps. The definitions and study of inward and weakly inward maps was begun by Halpern [8, 9]. The condition (4) was considered by Browder and Petryshyn [4]. Shoenberg considered (5) in [18]. From the following Corollaries 4–8 we do extend the above theorems, considered by various authors mentioned above, in Hilbert space.

COROLLARY 4. *Let S be a nonempty closed convex subset of a Hilbert space X, f either a continuous densifying map or a nonexpansive map of S into X. If $f(S)$ is bounded and f satisfies any one of the five conditions of Theorem 5, then f has a fixed point in S.*

Proof. Following the same proof as that of Corollaries 1 and 2, we can show that $(I\text{-pof})(\text{cl co pof}(S))$ is closed in X, where p is the proximity map of X into S. From Theorem 5, f has a fixed point in S.

Remark. Lin [11] and Singh and Watson [19] gave partial results of Corollary 4 for a continuous densifying map and a nonexpansive map, respectively.

COROLLARY 5. *Let S be a nonempty closed convex subset of a Hilbert space X, f a continuous semicontractive map of S into X. If $f(S)$ is bounded and f satisfies any one of the five conditions of Theorem 5, then f has a fixed point in S.*

Proof. Following the same proof as that of Theorem 2, $(I\text{-pof})(\text{cl co pof}(S))$ is closed. Since f is 1-set-contractive [14], from Theorem 5, f has a fixed point in S.

COROLLARY 6. *Let S be a nonempty closed convex subset of a Hilbert space X, g a nonexpansive map from S into X, and h a complete continuous map from S into X. If $f = g + h$, $f(S)$ is bounded, and f satisfies any one of the five conditions of Theorem 5, then f has a fixed point in S.*

This is just a corollary of Corollary 5.

COROLLARY 7. *Let S be a nonempty closed bounded convex subset of a Hilbert space X, f a* LANE *map of S into X. If f satisfies any one of the five conditions of Theorem 5, then f has a fixed point in S.*

Proof. From the proof of Theorem 3, (I-pof)(S) is closed in X, where p is the proximity map of X into S. From Theorem 5, f has a fixed point in S.

COROLLARY 8. *Let S be a nonempty closed bounded convex subset of a Hilbert space X, g a* LANE *map of S into X, and h a complete continuous map of S into X. If $f = g + h$, and f satisfies any one of the five conditions of* Theorem 5, *then f has a fixed point in S.*

Proof. Since f is also a LANE map [14, Remark 3.7], this corollary follows from Corollary 7.

REFERENCES

1. F. E. BROWDER, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.* **74** (1968), 660–665.
2. F. E. BROWDER, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *in* "Proc. of Symposia in Pure Math.," Vol. 18, Part 2, Amer. Math. Soc. Providence, RI, 1976.
3. F. E. BROWDER, On a sharpened form of the Schauder fixed-point theorem, *Proc. Natl. Acad. Sci. U.S.A.* **74** No. 11 (1977), 4749–4751.
4. F. E. BROWDER AND W. V. PETRYSHYN, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* **20** (1967), 197–228.
5. W. Cheney and A. H. Goldstein, Proximity maps for convex sets, *Proc. Amer. Math. Soc.* **10** (1959), 448–450.
6. KY FAN, Extensions of two fixed point theorems of F. E. Browder, *Math. Z.* **112** (1969), 234–240.
7. M. FURI AND A. VIGNOLI, A fixed point theorem in complete metric spaces, *Bull. Un. Math. Ital.* (*4*) **2** (1969), 505–509.
8. B. HALPERN, Fixed point theorems for set-valued maps in infinite dimensional spaces, *Math. Ann.* **189** (1970), 87–98.
9. B. HALPERN AND G. M. BERGMAN, A fixed point theorem for inward and outward maps, *Trans. Amer. Math. Soc.* **130** (1968), 353–358.
10. C. KURATOWSKI, Sur les espaces completes, *Fund. Math.* **15** (1930), 301–309.
11. T. C. LIN, A note on a theorem of Ky Fan, *Canad. Math. Bull.* **22** (4) (1979), 513–515.
12. R. D. NUSSBAUM, "The Fixed Point Index and Fixed Point Theorems for k-Set-Contractions," Doctoral dissertation, The Univ. of Chicago, 1969.
13. W. V. PETRYSHYN, Structure of the fixed points sets of K-set-contractions, *Arch. Ration. Mech. Anal.* **40** (1971), 312–328.
14. W. V. PETRYSHYN, Fixed point theorems for various classes of 1-set-contractive and 1-ball-contractive mappings in Banach spaces, *Trans. Amer. Math. Soc.* **182** (1973), 323–352.
15. S. REICH, Fixed points of condensing functions, *J. Math. Anal. Appl.* **41** (1973), 460–467.
16. S. REICH, Fixed points of nonexpansive functions, *J. London Math. Soc.* (*2*) **7** (1973), 5–10.
17. B. N. SADOVSKI, A fixed point principle, *Func. Anal. Appl.* **1** (1967), 151–153.
18. R. SCHONEBERG, Some fixed point theorems for mappings of nonexpansive type, *Comment. Math. Univ. Carolin.* **17** (1976), 399–411.
19. S. P. SINGH AND B. WATSON, Proximity maps and fixed points, *J. Approx. Theory* **39** (1983), 72–76.